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ABSTRACT

An analytical study was conducted for a semi-infinite flat plate that had fluid impulsively moved on its surface and at the same time pulsed with uniform heat flux over a prescribed length. The numerical solution, which was obtained by an implicit finite-difference method with nonuniform lattice dimensions, was compared with a solution for transient conduction in a slab and with a solution of the complete energy equation from the integral method of Karman-Pohlhausen. The solutions were confined to positions on the plate over which the flow is strictly transient and to fluids with Prandtl numbers greater than 1.

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SUMMARY

An analytical study was conducted for a semi-infinite flat plate that had fluid impulsively moved on its surface and at the same time pulsed with uniform heat flux over a prescribed length. The numerical solution, which was obtained by an implicit finite-difference method with nonuniform lattice dimensions, was compared with a solution for transient conduction in a slab and with a solution of the complete energy equation from the integral method of Karman-Pohlhausen. The solutions were confined to positions on the plate over which the flow is strictly transient and to fluids with Prandtl numbers greater than 1.

INTRODUCTION

In recent years, transient forced-convection heat-transfer has become of increasing interest to scientists and engineers. Solutions for this class of problems have applications to devices, such as the rocket engine and the nuclear reactor, in which startup and shutdown are important phases in the operation cycle of the equipment. The need for solutions to transient forced-convection problems may also be necessitated by experimental testing in facilities that afford relatively short testing times, such as the shock tunnel and the zero-gravity drop tower. In the former, "pulse type" testing is conducted while in the latter, experiments may not be initiated until after they are released.

Transient forced-convection heat-transfer problems may be classified as external (flat plate) or internal (pipe) flows. A further breakdown denotes the transient as hydrodynamic, thermal, or both hydrodynamic and thermal. In the hydrodynamic problem, thermal boundary conditions are not a function of time and the fluid flow is unsteady. Transient thermal problems are characterized by constant fluid flow, and the wall heat flux or the wall temperature are a prescribed function of time. The third case, of course, treats both unsteady flow and unsteady thermal boundary conditions. A survey of the work in this field has been provided by Soliman (ref. 1). Recent work includes references 2 and 3.

The purpose of this report is to solve an external problem that is unsteady both hydrodynamically and thermally. The problem may be described as follows: A flat plate is initially in thermal equilibrium with quiescent fluid that is above it. At some pre-

scribed time, the fluid is impulsively set in motion, and the plate receives a step increase in heat flux over a portion of its surface. Rozenshtok in a recent paper (ref. 4) treated a similar problem except that he assumed a step increase in surface temperature rather than in heat flux. He also did not consider the case for an unheated entrance length. In the present work, the problem is solved analytically by use of the integral method of Karman-Pohlhausen. These solutions are compared with a numerical solution which is obtained by use of an implicit finite-difference method with nonuniform lattice dimensions and which is assumed to best approximate true conditions. The validity of this assumption is substantiated by the close agreement between the numerical solution and an analytical solution for transient conduction in a slab for times and positions on the plate for which conduction is the heat-transfer mechanism. The solution is confined to positions on the plate over which the flow is strictly transient and to fluids with Prandtl numbers greater than 1.

ANALYSIS

The equations to be solved for transient laminar flow in a thermal boundary layer are

Continuity:

$$\frac{\partial U}{\partial \bar{x}} + \frac{\partial V}{\partial y} = 0 \tag{1}$$

Momentum:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial \bar{x}} + V\frac{\partial U}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2} - \frac{\nu}{\mu}\frac{dP}{d\bar{x}} \tag{2}$$

Energy:

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial \bar{x}} + V\frac{\partial T}{\partial y} = \frac{\nu}{Pr}\left(\frac{\partial^2 T}{\partial y^2}\right) + \frac{\nu}{C_p}\left(\frac{\partial U}{\partial y}\right)^2 \tag{3}$$

where the symbols are defined in appendix A. The fluid is assumed to be incompressible and the viscosity constant. Because of these restrictions, the continuity and momentum equations are independent of the energy equation and, therefore, may be solved directly for the velocity components U and V.

Fluid Flow Conditions

The fluid flow problem associated with impulsive motion on a flat plate has been treated by numerous investigators, references 5 and 6 being representative. Their solutions require simplification of the governing equations and are of two types. One result is based on a linearization of the momentum equation to an Oseen-type (Rayleigh's method) and another utilizes the momentum-integral method (Karman-Pohlhausen solution). Both solutions indicate that there are two domains on the plate that are not analytically joined. In the first domain, the velocity profile is solely a function of time; in the second domain, the velocity profile is only dependent on distance from the leading edge of the plate. A primary difference between the two results is the prediction of the extent of the domains. The Rayleigh solution indicates that it occurs at $\bar{x} = U_\infty t$ (ref. 5), whereas the momentum-integral result shows that it occurs at $\bar{x} = 0.385\ U_\infty t$ (ref. 6). All symbols are defined in appendix A.

With a knowledge of the aforementioned work, Stewartson (ref. 7) treated the full time-dependent momentum equations and obtained solutions in the form of power series. His results indicate that for $\bar{x} > U_\infty t$ the time-dependent Rayleigh solution is the correct solution and that for $\bar{x} < 0.377\ U_\infty t$ the position-dependent momentum-integral solution is the appropriate form to use. The domain between these two, $U_\infty t > \bar{x} > 0.377\ U_\infty t$, is a transition region in which the velocity is dependent on both time and position. These results are shown graphically in figure 1. A detailed discussion of Stewartson's work is contained in reference 8.

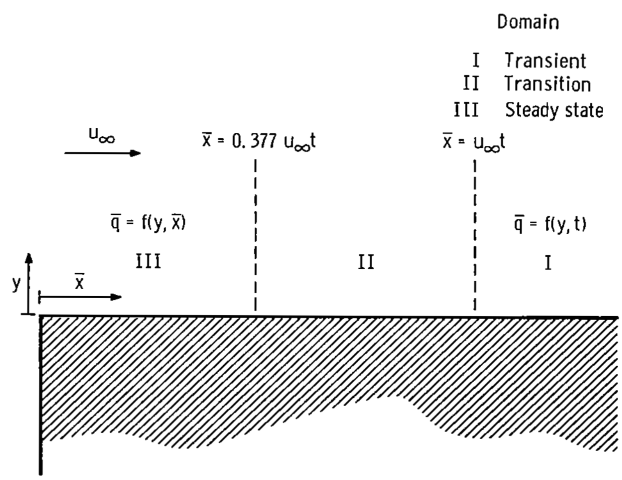


Figure 1. - Hydrodynamic conditions for impulsive motion on flat plate.

The present work is confined to those positions on the plate for which $\bar{x} > U_\infty t$. Therefore, an exact solution for the velocity profile is available and is

$$V = 0 \tag{4}$$

$$U = U_\infty \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) \tag{5}$$

However, for the purpose of obtaining an analytical solution for the energy equation, the velocity should be in a form that can be conveniently integrated. An expression in reference 4 that closely approximates equation (5) is the fourth-degree polynomial

$$U = U_\infty\left[2\left(\frac{y}{\delta_H}\right) - 2\left(\frac{y}{\delta_H}\right)^3 + \left(\frac{y}{\delta_H}\right)^4\right] \tag{6}$$

where $\delta_H = 3.65\sqrt{\nu_t}$. A comparison between equations (5) and (6) is made in figure 2.

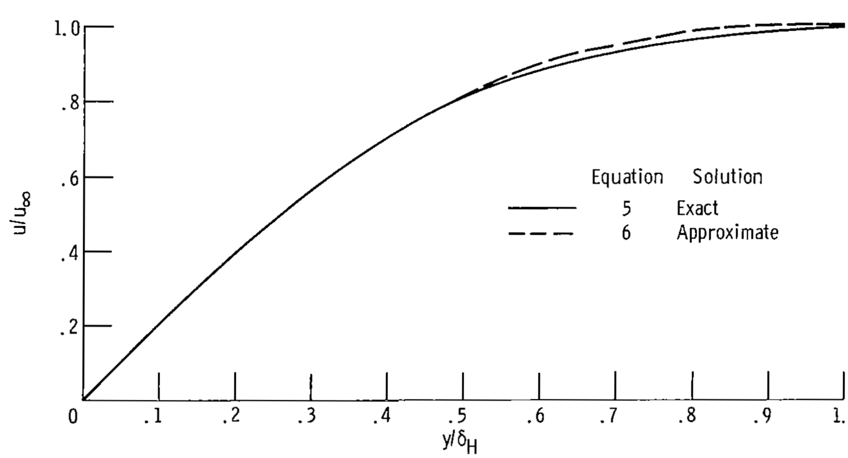


Figure 2. - Approximate and exact transient velocity profiles for impulsive motion on semi-infinite flat plate.

Thermal Conditions

The plan to be followed in finding a solution for the thermal problem is to approach it from both an analytical and a numerical viewpoint. The primary analytical approach used is the integral method whereas an implicit finite-difference method was used to find the numerical solution. A comparison of the solutions from the different methods can then be made.

Integral solution. - The thermal problem to be solved is that of a semi-infinite flat plate subjected to liquid impulsively set in motion in the plane of the plate that at the same time is pulsed with a uniform heat flux over a portion of its surface (i.e., the plate has an unheated entrance length $\overline{x}_0$). Attention is confined to those positions in the heated region for which the initial hydrodynamic transition $\overline{x} = U_\infty t$ is upstream.

The energy equation (eq. (3)), when viscous dissipation is neglected and equation (4) is applied, becomes

$$\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial \overline{x}} = \frac{\nu}{Pr} \frac{\partial^2 T}{\partial y^2} \tag{7}$$

The boundary conditions are

$$T = T_\infty \qquad \text{at } (\overline{x}_0, y, t)$$

$$T = T_\infty \qquad \text{at } (\overline{x} > \overline{x}_0, \infty, t)$$

$$\frac{\partial T}{\partial y} = -\frac{Q_W}{Ak} \qquad \text{at } (\overline{x} > \overline{x}_0, 0, t)$$

and the initial condition is

$$T = T_\infty \qquad \text{at } (\overline{x} > \overline{x}_0, y, 0)$$

The problem is solvable by the integral method. Therefore, equation (7) becomes

$$\frac{\partial}{\partial t}\int_0^{\delta_T} (T - T_\infty)dy + \frac{\partial}{\partial \overline{x}}\int_0^{\delta_T} U(T - T_\infty)dy = -\frac{\nu}{Pr} \frac{\partial T}{\partial y}\bigg|_{y=0} \tag{8}$$

where δ_T is defined as that position above the surface of the plate where $T = T_\infty$ and Pr is the Prandtl number $\mu C_p/k$. Equation (8) is written for fluids with Prandtl numbers greater than 1, and the solution will be confined to this case.

The formulation may be further simplified by defining a new dependent variable

$$\theta = \frac{T - T_\infty}{\left(\frac{Q_W}{Ak}\right)}$$

The boundary and initial conditions become

$$\theta = 0 \text{ at } (\bar{x}_o, y, t),\ (\bar{x} > \bar{x}_o, \delta_T, t),\ \text{and } (\bar{x} > \bar{x}_o, y, 0)$$

$$\frac{\partial \theta}{\partial y} = -1 \text{ at } (\bar{x} > \bar{x}_o, 0, t)$$

and equation (8) is simplified to

$$\frac{\partial}{\partial t}\int_0^{\delta_T} \theta\, dy + \frac{\partial}{\partial \bar{x}}\int_0^{\delta_T} U\theta\, dy = \frac{\nu}{Pr} \tag{9}$$

The dependent variable is now approximated by a polynomial in y. An expression that has been used by others working on transient thermal problems (see refs. 9 and 10) and has yielded acceptable results is

$$\theta = \frac{\delta_T}{2}\left(1 - \frac{y}{\delta_T}\right)^2 \tag{10}$$

Substitution of equations (6) and (10) into equation (9) yields

$$\frac{1}{6}\frac{\partial \delta_T{}^2}{\partial t} + U_\infty \frac{\partial}{\partial \bar{x}}\left[\delta_T{}^2\left(\frac{1}{12}E - \frac{1}{60}E^3 + \frac{1}{210}E^4\right)\right] = \frac{\nu}{Pr} \tag{11}$$

where $E = \delta_T/\delta_H$. For the particular case at hand, $E \leq 1$ so that the term in parenthesis may be approximated by an arithmetic average with little error:

$$\frac{1}{6}\frac{\partial(\delta_T{}^2)}{\partial t} + \frac{13}{168}\left(\frac{U_\infty}{\delta_H}\right)\frac{\partial(\delta_T{}^3)}{\partial \bar{x}} = \frac{\nu}{Pr} \tag{12}$$

This equation may be simplified by letting

$$\delta_T{}^2 = \psi$$

so that

$$\frac{1}{6}\frac{Pr}{\nu}\frac{\partial\psi}{\partial t} + \frac{13}{112}\frac{U_\infty P_r}{\nu\,\delta_H}(\psi)^{1/2}\frac{\partial\psi}{\partial\bar{x}} = 1 \tag{13}$$

Equation (13) is solvable by the method of characteristics, the details of which are presented in reference 11. The system of equations for the characteristics of (13) may be written as

$$6\left(\frac{\nu}{Pr}\right)dt = \frac{112}{13}\left(\frac{\nu\,\delta_H}{U_\infty Pr\psi^{1/2}}\right)d\bar{x} = d\psi \tag{14}$$

The solutions of the system (14) are

$$\delta_T = 2.45\sqrt{\frac{\nu t}{Pr}} \tag{15}$$

$$\delta_T = 2.04\left[\frac{\nu\,\delta_H(\bar{x} - \bar{x}_o)}{U_\infty Pr}\right]^{1/3} \tag{16}$$

$$\bar{x} - \bar{x}_o = \left[\frac{0.468}{(Pr)^{1/2}}\right]U_\infty t \tag{17}$$

Substitution of equations (15) and (16) into equation (10) results, respectively, in

$$\theta = 1.225\sqrt{\frac{\nu t}{Pr}}\left(1 - \frac{0.408\ y}{\sqrt{\frac{\nu t}{Pr}}}\right)^2 \tag{18}$$

$$\theta = 1.02\left[\frac{\nu\,\delta_H(\bar{x} - \bar{x}_o)}{U_\infty Pr}\right]^{1/3}\left\{1 - \frac{0.490\,y}{\left[\frac{\nu\,\delta_H(\bar{x} - \bar{x}_o)}{U_\infty Pr}\right]^{1/3}}\right\}^2 \tag{19}$$

The solution predicts two regimes on the plate that are not analytically joined. The temperature in one (eq. (18)) is dependent only on time (conduction regime) while in the other (eq. (19)) it is dependent on distance from the thermal leading edge, time, and free-stream velocity (convection regime).

The analysis indicates that the boundary between the conduction and convection regimes is described by equation (17). However, in light of Stewartson's hydrodynamic analyses, it seems reasonable to assume that there is a transition regime between these two regimes. For this particular solution, equation (17) describes the upper extent of the convection regime whereas a conservative estimate of the lower limit of the conduction regime is $\bar{x} - \bar{x}_o = U_\infty t$. A better estimate for this limit is determined later in the section Extent of Regimes. The conditions existing on the plate are shown graphically in figure 3 where the extent of the regimes is denoted in a general form.

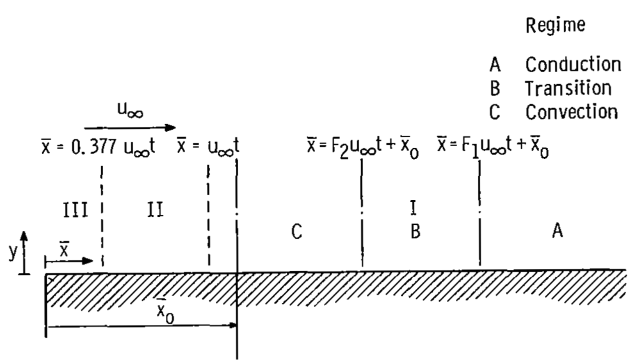


Figure 3. - Thermal conditions for impulsive motion on flat plate with unheated entrance length.

Since the solution obtained by this method depends on the form of the polynomial that was assumed for θ in terms of y (see eq. (10)), other solutions to the problem are possible by assuming different polynomials. A form in common use besides the parabolic profile is the quartic profile (ref. 4). For the problem at hand, the boundary conditions necessary to form the required equation are

$$\theta = 0 \quad \text{at } y = \delta_T \qquad \frac{\partial\theta}{\partial y} = 0 \quad \text{at } y = \delta_T$$

$$\frac{\partial\theta}{\partial y} = -1 \quad \text{at } y = 0 \qquad \frac{\partial^2\theta}{\partial y^2} = 0 \quad \text{at } y = 0$$

$$\frac{\partial^2\theta}{\partial y^2} = 0 \quad \text{at } y = \delta_T$$

The quartic polynomial is

$$\theta = \left(\frac{1}{2}\right)\delta_T - y + \left(\frac{1}{\delta_T{}^2}\right)y^3 - \left(\frac{1}{\delta_T{}^3}\right)y^4 \tag{20}$$

and the solutions to the problem are

$$\theta = 1.290\sqrt{\frac{\nu t}{Pr}} - y + 0.150\left(\frac{Pr}{\nu t}\right)y^3 - 0.0291\left(\frac{Pr}{\nu t}\right)^{3/2}y^4 \tag{21}$$

$$\theta = 1.10\left[\frac{\delta_H \nu(\bar{x} - \bar{x}_o)}{U_\infty Pr}\right]^{1/3} - y + 0.207\left[\frac{U_\infty Pr}{\delta_H \nu(\bar{x} - \bar{x}_o)}\right]^{2/3} y^3 - 0.047\left[\frac{U_\infty Pr}{\nu\,\delta_H(\bar{x} - \bar{x}_o)}\right]y^4 \tag{22}$$

$$\bar{x} - \bar{x}_o = \left[\frac{0.443}{(Pr)^{1/2}}\right]U_\infty t \tag{23}$$

Numerical solution. - The form of the energy equation to be solved numerically is

$$\frac{\partial\theta}{\partial t} + U_\infty \operatorname{erf}\eta\,\frac{\partial\theta}{\partial x} = \alpha\,\frac{\partial^2\theta}{\partial y^2} \tag{24}$$

where $\eta = y/2\sqrt{\nu t}$, α is the thermal diffusivity and $x = \bar{x} - \bar{x}_o$. An approximate solution of equation (24), with its initial and boundary conditions, was found using an implicit finite-difference method. The partial differential equation was replaced by a finite-difference equation which was then solved. This process is equivalent to finding $\theta(x, y, t)$ at discrete lattice points in the x-, y-, t-space (fig. 4). Because of the exponential nature

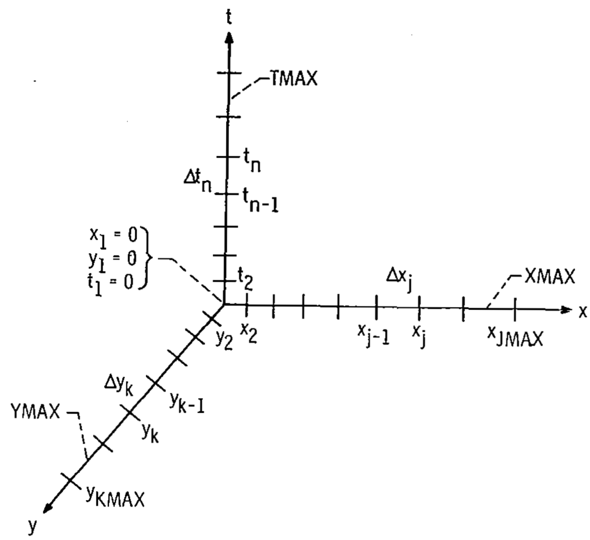


Figure 4. - Lattice dimensions in x, y, t-space.

of the solution, and our interest near the origin, nonuniform lattice dimensions were used. These dimensions may be expressed as

$$\Delta t_n = t_n - t_{n-1} = \mathrm{PTINT}(\Delta t_{n-1})$$

$$\Delta x_j = x_j - x_{j-1} = \mathrm{PXINT}(\Delta x_{j-1})$$

$$\Delta y_k = y_k - y_{k-1} = \mathrm{PYINT}(\Delta y_{k-1})$$

where PTINT, PXINT, and PYINT are greater than or equal to one.

The numerical values of PTINT, PXINT, PYINT, Δt_2, Δx_2, Δy_2, KMAX, and JMAX were determined by trial and error under the condition that the solution not be affected by lattice sizes, that x_{JMAX} be greater than or equal to the heated length of the plate, and that y_{KMAX} be greater than or equal to the maximum estimated thermal layer thickness.

The finite-difference equivalent of equation (24), as derived in appendix B, is

$$\theta_{j,k-1,n}(-\rho) + \theta_{j,k,n}\left[\beta + \gamma + \rho(1 + \epsilon)\right] + \theta_{j,k+1,n}(-\rho\epsilon) = \beta\theta_{j,k,n-1} + \gamma\theta_{j-1,k,n} \tag{25}$$

where ϵ, β, γ, and ρ are defined in appendix A, and for simplicity $\theta(x_j, y_k, t_n)$ is expressed as $\theta_{j,k,n}$. Initial and boundary conditions are

$$\theta_{j,k,1} = 0 \qquad \text{over all } j, k$$

$$\theta_{1,k,n} = 0 \qquad \text{over all } k, n$$

$$\theta_{j,KMAX+1,n} = 0 \qquad \text{over all } j, n$$

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = \frac{1}{\Delta y_2}\left(\theta_{j,2,n} - \theta_{j,1,n}\right) = -1 \qquad j > 1,\ n > 1$$

If equation (25) is written for all lattice points of $y > 0$ at fixed values of x and t

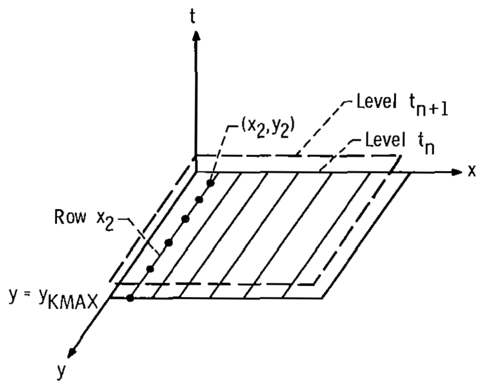


Figure 5. - One row of lattice points.

(see fig. 5), a system of KMAX - 1 number of linear equations with KMAX - 1 unknowns is obtained:

$$\begin{aligned}
b_1Z_1 + c_1Z_2 \qquad\qquad &= d_1 \\
a_2Z_1 + b_2Z_2 + c_2Z_3 \qquad &= d_2 \\
a_3Z_2 + b_3Z_3 + c_3Z_4 &= d_3 \\
\cdot \qquad &\quad \cdot \\
\cdot \qquad &\quad \cdot \\
\cdot \qquad &\quad \cdot \\
a_mZ_{m-1} + b_mZ_m &= d_m
\end{aligned} \tag{26}$$

where $Z_i = \theta_{j,i+1,n}$, $m = \mathrm{KMAX} - 1$, and a, b, c, and d are defined in appendix B. This system of equations is a tridiagonal matrix, all elements of which are zero except for the main diagonal and the two diagonals on either side of it. The coefficients of Z, the values for which may be found in appendix B, are functions of lattice point positions. At the first assumed time $t = t_2$ and displacement $x = x_2$, the elements on the right side of the system (26) are determined by the initial and boundary conditions. The variable θ is now determined from the matrix by Gauss' elimination (see ref. 12). The solution of the matrix is presented in appendix C. With θ known for the first x-row of lattice points, another set of KMAX - 1 equations can be written for the next row of lattice points in the same time plane. The elements on the right side of the matrix for this solution are obtained from the results of the solution from the preceeding x-row. This process is continued until the maximum displacement in x is reached. After obtaining the solution for all lattice points in the x, y-plane at the first time, $\theta_{j,k,n}$ can now be found at successive values of time by repeating the above processes until the time reaches the limit of interest, TMAX.

A FORTRAN IV computer program was written to find the solution of the finite-difference equation in the manner just described. A description of the computer program with flow charts and a listing of the programs is given in appendix D.

Transient conduction solution. - The integral solution indicates that for some positions on the plate the mode of heat transfer is strictly conduction. This suggests that for such positions a solution may be obtained from the energy equation in the form

$$\frac{\partial T}{\partial t} = \left(\frac{\nu}{Pr}\right)\frac{\partial^2 T}{\partial y^2} \tag{27}$$

with the boundary conditions

$$T = T_\infty \quad \text{at } (y, 0)$$

$$T = T_\infty \quad \text{at } (\infty, t)$$

$$\frac{\partial T}{\partial y} = -\frac{Q_W}{Ak} \quad \text{at } (0, t)$$

This problem has been solved in reference 13 and the solution is

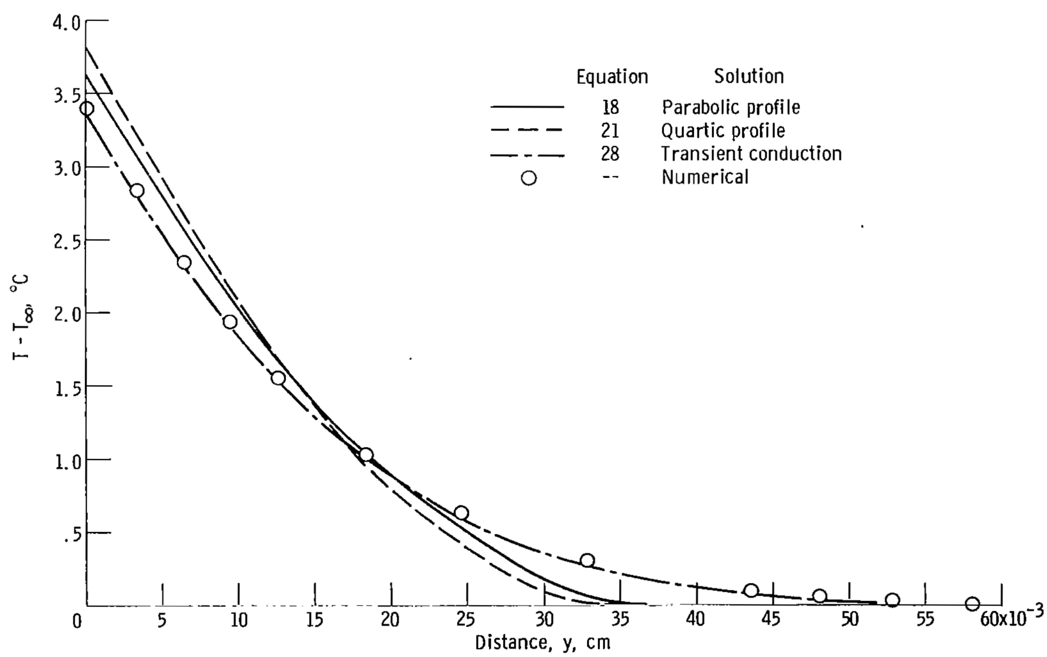
$$\theta = 2\sqrt{\frac{\nu t}{\Pr}}\ \text{ierfc}\left(\frac{y}{2\sqrt{\frac{\nu t}{\Pr}}}\right) \tag{28}$$

RESULTS

Conduction and Convection Regimes

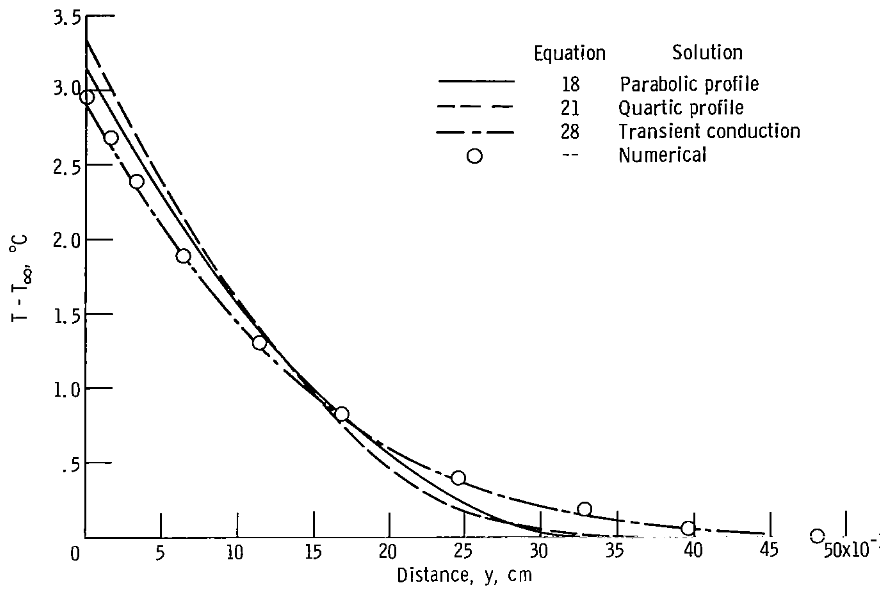
A comparison between the analytical and numerical solutions was made for water as a working liquid. The conditions assumed were a free-stream temperature of 98.9° C and free-stream velocities of 3.05 and 12.2 centimeters per second. The surface heat flux was taken as 1261 watts per square meter. In comparing the results, the numerical solution was assumed to always best approximate the true conditions.

The validity of the conduction equations was tested by assuming velocities, times, and positions such that $\bar{x} - \bar{x}_0 > U_\infty t$. The results are shown in figures 6(a) and (b). The curves indicate that, of the integral solutions, the parabolic profile is in best overall

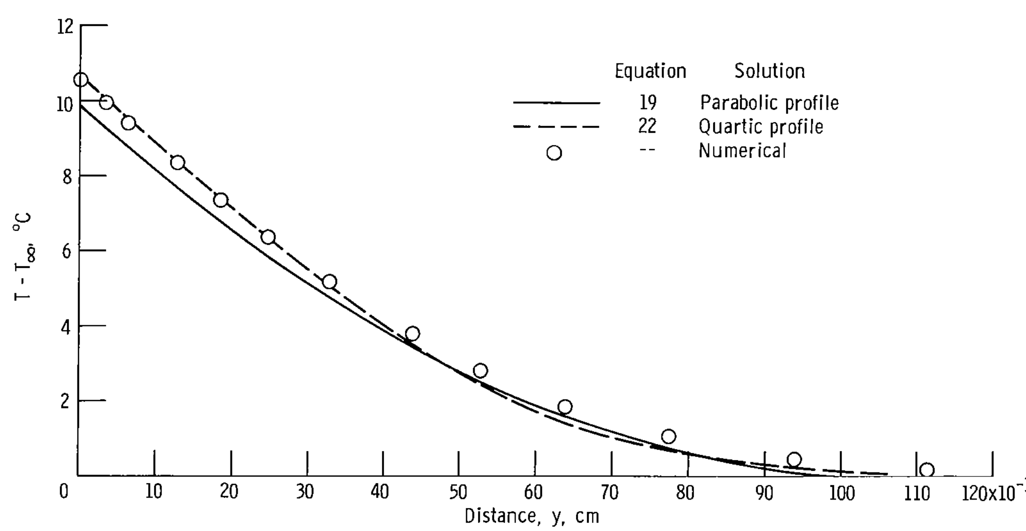


(a) Free-stream velocity in x-direction, 3.05 centimeters per second; distance from leading thermal edge, 0.915 centimeter; time, 0.154 second.

Figure 6. - Temperature profiles.

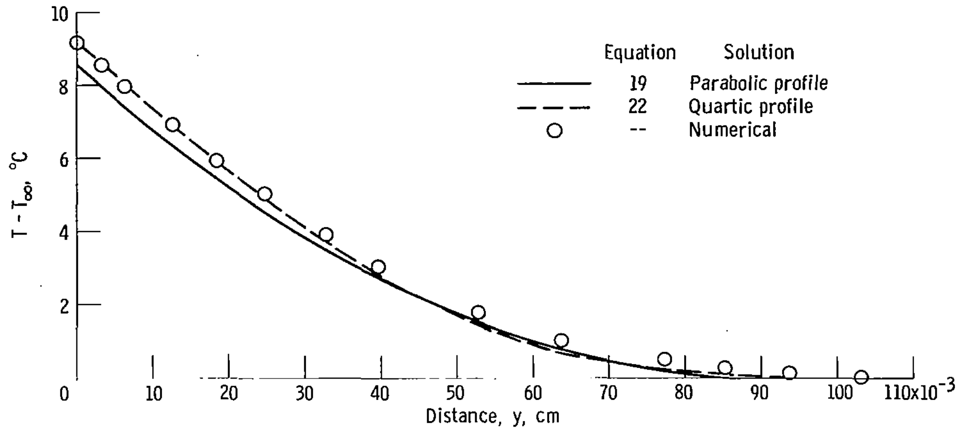


(b) Free-stream velocity in x-direction, 12.2 centimeters per second; distance from leading thermal edge, 2.72 centimeters; time, 0.116 second.



(c) Free-stream velocity in x-direction, 3.05 centimeters per second; distance from leading thermal edge, 0.915 centimeter; time, 2.02 seconds.

Figure 6. - Continued.



(d) Free-stream velocity in x-direction, 12.2 centimeters per second; distance from leading thermal edge, 2.72 centimeters; time, 1.52 seconds.

Figure 6. - Concluded.

agreement with the numerical solution, the greatest discrepancy between the two occurring at large distances from the surface. This is understandable in light of the finite boundary condition imposed on the integral solution at large y as opposed to the infinite one that actually exists. The close agreement between the transient conduction solution and the numerical solution is expected and validates the numerical approach to the problem.

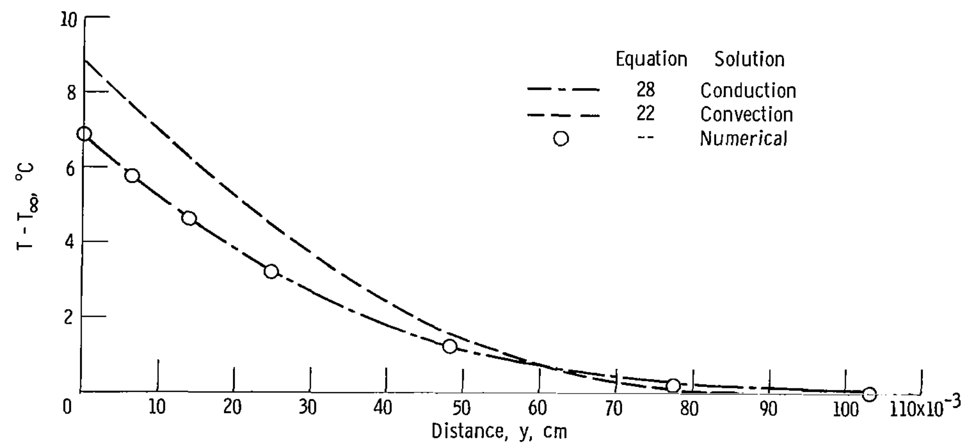
The convection equations were evaluated for conditions so that $\bar{x} - \bar{x}_0 < 0.324\ U_\infty t$ (eq. (23)). This criterion was chosen from the two available ones (eqs. (17) and (23)) because it minimized the size of the convection regime, thus ensuring that the selected conditions were definitely out of the transition regime. The results in figures 6(c) and (d) show that the quartic profile is in the best overall agreement with the numerical solution.

In summary, for the conduction regime, the transient conduction solution (eq. (28)) is representative of the temperature profile. For the convection regime, the integral solution using a quartic profile (eq. (22)) provides a good approximation for the temperature distribution.

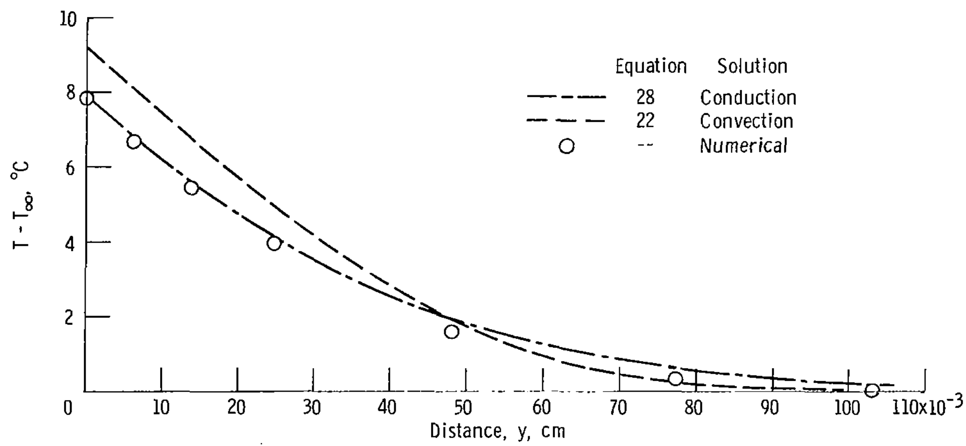
Transition Regime

The nature of the temperature distribution between the conduction and convection regimes was investigated by selecting a velocity (3.05 cm/sec) and a position (0.915 cm) and determining the numerical solution for different times. The results are presented in figure 7 where the conduction and convection solutions, as determined in the previous section, are also plotted.

The conditions in figure 7(a) are such that the position selected is just out of the conduction regime, as defined by the extent criteria $\infty > \overline{x} - \overline{x}_0 > U_\infty t$. As seen from the figure, the conduction solution is still generally in good agreement with the numerical solution. However, at large distances from the surface, the two begin to deviate from one another, signaling the start of convection effects. This result may be logically obtained by considering the physical processes that are occurring. Conduction dominates as the mode of heat transfer at a position in the thermal boundary layer until fluid that was originally upstream of the thermal leading edge reaches the position in question. Since the liquid velocity is greater at large distances from the surface, the convection ef-



(a) Free-stream velocity in x-direction, 3.05 centimeters per second; distance from leading thermal edge, 0.915 centimeter, time, 0.643 second.

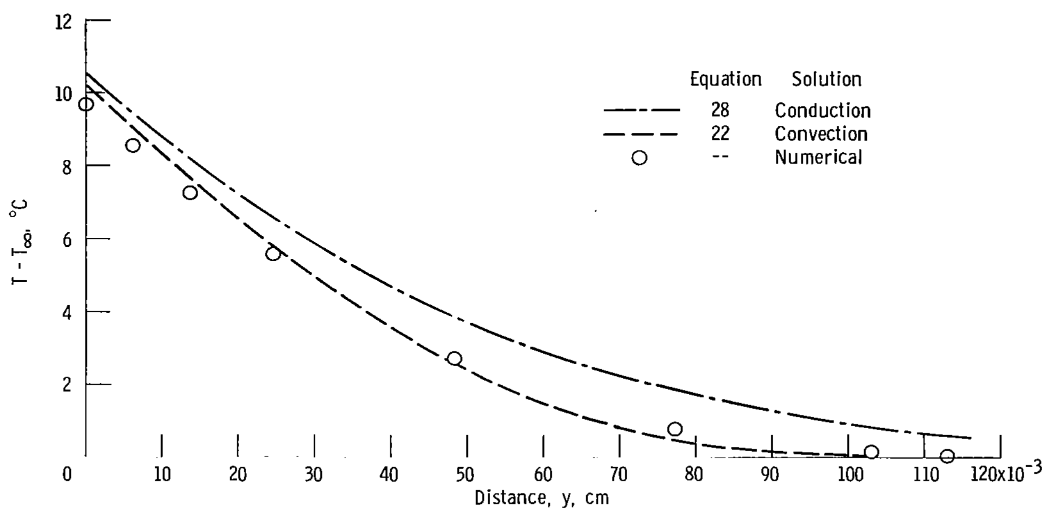


(b) Free-stream velocity in x-direction, 3.05 centimeters per second; distance from leading thermal edge, 0.915 centimeter; time, 0.856 second.

Figure 7. - Temperature profiles.

fects are realized first at those distances.

The results in figure 7(b) are for a later time than those in figure 7(a). Here again, the conduction solution is in fairly good agreement with the numerical solution; however, the deviation between these two solutions at large distances from the plate is greater than that in figure 7(a). Comparison of the convection curve in figures 7(a) and (b) indicates, that as time progresses, it comes into closer agreement with the numerical solution. It is obvious then, that the importance of convection as a mode of heat transfer increases with time.



(c) Free-stream velocity in x-direction, 3.05 centimeters per second; distance from leading thermal edge, 0.915 centimeter; time, 1.516 second.

Figure 7. - Concluded.

For figure 7(c), the conditions are such that the selected position is just in the convection regime as determined by the extent criterion $0 < \bar{x} - \bar{x}_0 < 0.324\ U_\infty t$. The curves indicate that the convection solution is now in best agreement with the numerical solution and that the transition from conduction to convection as the mode of heat transfer has penetrated to small distances from the surface of the plate.

Two primary conclusions are to be drawn from these results: The first is that during this regime the mode of heat transfer is changed from conduction to convection. The second conclusion is that in the transition regime the conduction solution of equation (28) is a good approximation for the temperature distribution.

Extent of Regimes

From the analysis, it was determined that the upper extent of the convection regime was defined by equation (23). The function F_2 in the extent criterion, as shown in figure 3, may then be expressed as $F_2 = 0.443/(Pr)^{1/2}$. For the conduction regime, its lower extent has been conservatively estimated as $\bar{x} - \bar{x}_0 = U_\infty t$. As discussed in the previous section, the onset of convection effects at a position $(\bar{x} - \bar{x}_0)$ in the thermal boundary layer is signaled by the arrival at the position of liquid that was originally upstream of the leading thermal edge. Since the velocity boundary layer is larger than the thermal boundary layer for fluids with Prandtl numbers greater than 1, the liquid in transit from the leading thermal edge to a position is not continuously moving at the free-stream velocity. Therefore, $\bar{x} - \bar{x}_0 = U_\infty t$ is conservative in that it overestimates the size of the transition regime.

A criterion for the lower extent of the conduction regime may be obtained by considering liquid particles in laminar flow at some distance y above the surface of the plate. An expression for the rectilinear motion of these particles is

$$\bar{x} - \bar{x}_0 = \bar{x}_\infty + \int_{t_\infty}^{t} U\,dt \tag{29}$$

where x_∞ denotes the displacement over which the liquid particles moved at the free-stream velocity and t_∞ the time required to move this distance. Inserting equation (6) into equation (29) and integrating results in

$$\bar{x} - \bar{x}_0 = \bar{x}_\infty + U_\infty\left[\left(\frac{1.096\,y}{\nu^{1/2}}\right)\left(t^{1/2} - t_\infty^{1/2}\right) + \left(\frac{0.082\,y^3}{\nu^{3/2}}\right)\left(\frac{1}{t^{1/2}} - \frac{1}{t_\infty^{1/2}}\right) - \left(\frac{0.0056\,y^4}{\nu^2}\right)\left(\frac{1}{t} - \frac{1}{t_\infty}\right)\right] \tag{30}$$

The displacement $\bar{x}_\infty$ may be expressed as

$$\bar{x}_\infty = U_\infty t_\infty \tag{31}$$

where t_∞ is determined from the time required for the hydrodynamic boundary layer to increase in size to the distance y under consideration. A relation for the hydrodynamic boundary layer, as presented in the section Fluid Flow Conditions, is

$$\delta_H = 3.65(\nu t)^{1/2} \tag{32}$$

so that

$$\bar{x}_\infty = \frac{0.0752\ U_\infty y^2}{\nu} \tag{33}$$

The initial convection effect at a position $(\bar{x} - \bar{x}_o)$ is realized when liquid at a distance y reaches the position at the same time that the conduction thermal boundary layer has increased in size to y. If a criterion for the thermal boundary layer is assumed to be

$$\frac{T - T_\infty}{T_W - T_\infty} = 0.01$$

equation (28) may be solved such that

$$\delta_T \cong 3.2\left(\frac{\nu t}{Pr}\right)^{1/2} = y \tag{34}$$

Inserting equations (32) to (34) into equation (30) results in

$$\bar{x} - \bar{x}_o = \left[\frac{3.507}{(Pr)^{1/2}} + \frac{2.687}{(Pr)^{3/2}} - \frac{0.587}{(Pr)^2} - \frac{4.608}{Pr}\right]U_\infty t \tag{35}$$

where F_1 (see fig. 3) is the term in parentheses. A plot of the extent criteria of equations (23) and (35) is provided in figure 8.

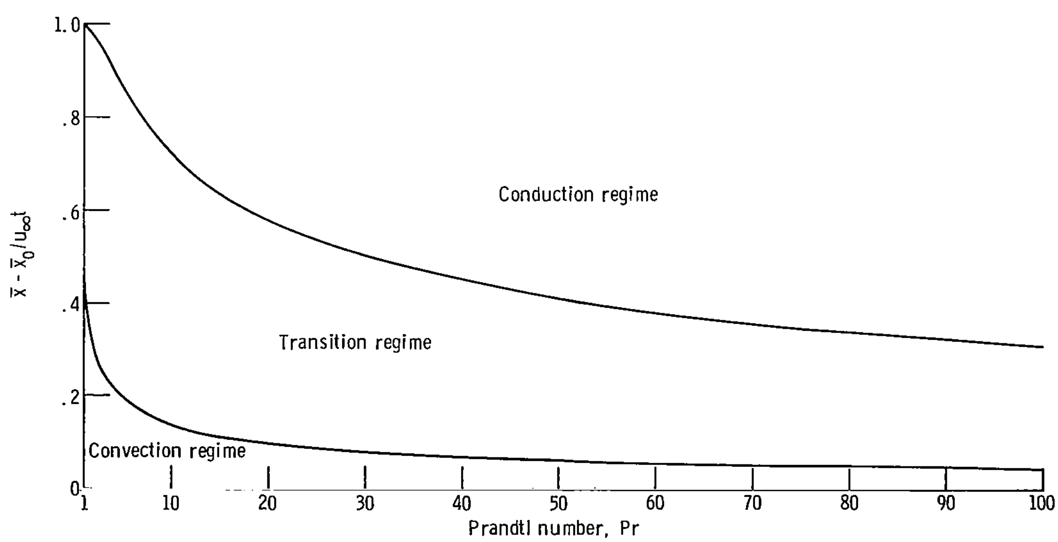


Figure 8. - Extent of regimes as function of Prandtl number.

SUMMARY OF RESULTS

An analytical and numerical study of the thermal conditions on a flat plate with an unheated entrance length, subjected to fluid impulsively moved, and pulsed with uniform heat flux yielded the following results:

1. For positions on the plate denoted as the conduction regime and describable by the conditions $\bar{x} > U_\infty t$ and $F_1 U_\infty t + \bar{x} < \bar{x} < \infty$, where $\bar{x}$ is the distance measured from the leading edge along the plate, U_∞ is the free stream velocity, t is time, F_1 is a function of the Prandtl number that is characteristic of the lower limit of the conduction regime, and $\bar{x}_o$ is the unheated entrance length, the temperature profile may be presented as

$$T - T_\infty = \frac{2Q_W}{Ak}\left(\frac{\nu t}{Pr}\right)^{1/2} \text{ierfc}\left[\frac{y}{2\left(\frac{\nu t}{Pr}\right)^{1/2}}\right]$$

where

$$F_1 = \frac{3.507}{(Pr)^{1/2}} + \frac{2.687}{(Pr)^{3/2}} - \frac{0.587}{Pr^2} - \frac{4.608}{Pr} \qquad \text{for } Pr > 1$$

and T is the fluid temperature, T_∞ is the free stream temperature, Q_W is the surface heat flux, A is the surface area, k is the fluid thermal conductivity, ν is the kinematic viscosity, Pr is the Prandtl number, and y is the distance measured perpendicular to the plate.

2. For positions on the plate denoted as the transition regime and describable by the conditions $\bar{x} > U_\infty t$ and $F_2 U_\infty t + \bar{x}_o < \bar{x} < F_1 U_\infty t + \bar{x}_o$, where F_2 is a function of the Prandtl number that is characteristic of the upper limit of the convection regime, the temperature profile may be approximated as

$$T - T_\infty \cong \frac{2Q_W}{Ak}\left(\frac{\nu t}{Pr}\right)^{1/2} \text{ierfc}\left[\frac{y}{2\left(\frac{\nu t}{Pr}\right)^{1/2}}\right]$$

where

$$F_2 = \frac{0.443}{(Pr)^{1/2}} \qquad \text{for} \quad Pr > 1$$

3. For positions on the plate denoted as the convection regime and describable by the conditions $\bar{x} > U_\infty t$ and $\bar{x}_o < \bar{x} < F_2 U_\infty t + \bar{x}_o$, the temperature profile may be best presented as

$$T - T_\infty = \frac{Q_W}{Ak}\left\{1.10\left[\frac{\delta_H \nu(\bar{x} - \bar{x}_o)}{U_\infty Pr}\right]^{1/3} - y + 0.207\left[\frac{U_\infty Pr}{\delta_H \nu(\bar{x} - \bar{x}_o)}\right]^{2/3} y^3 - 0.047\left[\frac{U_\infty Pr}{\nu\, \delta_H(\bar{x} - \bar{x}_o)}\right] y^4\right\}$$

where δ_H is the hydrodynamic boundary layer thickness.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, February 27, 1969,
124-09-17-01-22.

APPENDIX A

SYMBOLS

A	heated area of plate, cm^2
a, b, c, d	array of constants in a tridiagonal matrix
C_p	specific heat, $J/(kg)(^oC)$
E	ratio of thermal to hydrodynamic boundary layer, δ_T/δ_H
F_1	function of Prandtl number that is characteristic of lower limit of conduction regime
F_2	function of Prandtl number that is characteristic of upper limit of convection regime
k	thermal conductivity, $J/(m)(sec)(^oC)$
m	order of matrix equation, KMAX - 1
P	pressure, N/m^2
Pr	Prandtl number, $\mu C_p/k$
PXINT	growth rate of x intervals, $\Delta x_j/\Delta x_{j-1}$
PYINT	growth rate of y intervals, $\Delta y_k/\Delta y_{k-1}$
PTINT	growth rate of t intervals, $\Delta t_n/\Delta t_{n-1}$
Q	heat flux, W
$\bar{q}$	total velocity, cm/sec
T	temperature, ^oC
TMAX	maximum time of interest
t	time, sec
U	velocity in x-direction, cm/sec
V	velocity in y-direction, cm/sec
XMAX	total heated length, cm
x	distance measured from leading thermal edge along plate, $x = \bar{x} - \bar{x}_o$, cm
$\bar{x}$	distance measured from leading edge along plate, cm
YMAX	maximum estimated thermal boundary layer thickness, cm
y	distance measured perpendicular to plate, cm

Z	unknown in matrix equation representing θ at all vertical points above plate at fixed t and x
α	thermal diffusivity, cm^2/sec
β	$1/\Delta t_n$
γ	U_∞ erf $\eta/\Delta x_j$
δ	boundary layer thickness, cm
ϵ	1/PYINT
η	argument of the error function, $y\Big/\left(2\sqrt{\nu t}\right)$
θ	dependent variable in energy equation, $\theta \equiv \dfrac{T - T_\infty}{Q_W/Ak}$
μ	dynamic viscosity, cP
ν	kinematic viscosity, $(m)^2/sec$
ρ	$\alpha/(\Delta y_k)^2$
ψ	$\delta_T 2$

Subscripts:

H	hydrodynamic conditions
JMAX	number of lattice points on x-scale
j, k, n	indices denoting specific lattice points on the x-, y-, and t-scales, respectively
KMAX	number of lattice points on y-scale
o	thermal entrance condition
T	thermal conditions
W	conditions at $y = 0$
∞	free-stream conditions

APPENDIX B

FINITE-DIFFERENCE APPROXIMATION

The finite-difference approximation of the energy equation in the form

$$\frac{\partial\theta}{\partial t} + U_\infty \mathrm{erf}(\eta)\,\frac{\partial\theta}{\partial x} = \alpha\,\frac{\partial^2\theta}{\partial y^2}$$

is derived in the following manner: From figure 4, it may be seen that

$$\left.\frac{\partial\theta}{\partial t}\right|_{j,k,n} = \frac{1}{\Delta t_n}\left(\theta_{j,k,n} - \theta_{j,k,n-1}\right)$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{j,k,n} = \frac{1}{\Delta x_j}\left(\theta_{j,k,n} - \theta_{j-1,k,n}\right)$$

The second partial derivative in y is obtained by considering two consecutive intervals

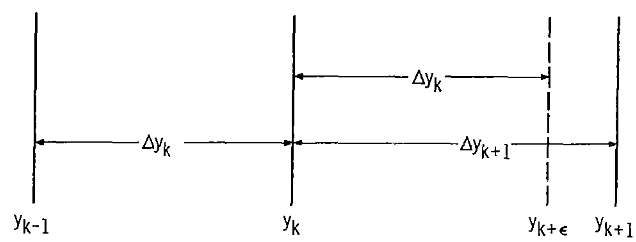


Figure 9. - Two intervals on y-scale.

on the y-scale (see fig. 9):

$$\left.\frac{\partial^2\theta}{\partial y^2}\right|_{j,k,n} = \left(\frac{1}{\Delta y_k}\right)^2\left(\theta_{j,k+\epsilon,n} - 2\theta_{j,k,n} + \theta_{j,k-1,n}\right)$$

where $\theta_{j,k+\epsilon,n}$ can be expressed by a linear interpolation, $\theta_{j,k+\epsilon,n} = \epsilon\theta_{j,k+1,n} + (1 - \epsilon)\theta_{j,k,n}$. Although the existence of $\partial^2\theta/\partial y^2$ proves that $\theta_{j,k+\epsilon,n}$ cannot be accurately approximated by a linear interpolation, for the case when $\epsilon = 0.91$, the results derived from the linear interpolation differs from that of a second-order approximation

by only 0.2 percent. The partial differential equation in finite-difference form is then

$$\frac{1}{\Delta t_n}(\theta_{j,k,n} - \theta_{j,k,n-1}) + \frac{U_\infty \operatorname{erf}(\eta)}{\Delta x_j}(\theta_{j,k,n} - \theta_{j-1,k,n})$$

$$= \frac{\alpha}{(\Delta y_k)^2}\left[\epsilon\theta_{j,k,+1,n} - (1+\epsilon)\theta_{j,k,n} + \theta_{j,k-1,n}\right] \qquad \text{(B1)}$$

with initial and boundary conditions

$$\theta_{j,k,1} = 0 \qquad j \geq 1, k \geq 1$$

$$\theta_{1,k,n} = 0 \qquad k \geq 1, n \geq 1$$

$$\theta_{j,KMAX+1,n} = 0 \qquad j \geq 1, n \geq 1$$

$$\frac{1}{\Delta y_2}(\theta_{j,2,n} - \theta_{j,1,n}) = -1 \qquad j > 1, n > 1$$

Equation (B1) can be rearranged in the form

$$\theta_{j,k-1,n}(-\rho) + \theta_{j,k,n}\left[\beta + \gamma + \rho(1+\epsilon)\right] + \theta_{j,k+1,n}(-\rho\epsilon) = \beta\theta_{j,k,n-1} + \gamma\theta_{j-1,k,n} \qquad \text{(B2)}$$

where $\rho, \beta, \gamma,$ and ϵ are defined in appendix A. Applying the boundary conditions at $k = 2$ yields

$$\theta_{j,k,n}(\beta + \gamma + \rho\epsilon) + \theta_{j,k+1,n}(-\rho\epsilon) = \beta\theta_{j,k,n-1} + \gamma\theta_{j-1,k,n} - \rho(\theta_{j,k,n} - \theta_{j,k-1,n}) \qquad \text{(B2a)}$$

The last term on the right side is $\rho(\Delta y_2)(-1)$. Applying the boundary condition at $k = KMAX$ gives

$$\theta_{j,k-1,n}(-\rho) + \theta_{j,k,n}\left[\beta + \gamma + \rho(1+\epsilon)\right] = \beta\theta_{j,k,n-1} + \gamma\theta_{j-1,k,n} \qquad \text{(B2b)}$$

For every given x_j and t_n, equations (B2) make up a system of KMAX - 1 equations with KMAX - 1 unknowns, $\theta_{j,k,n}(k = 2, 3, \ldots KMAX)$, in the following form:

$$
\begin{aligned}
&b_1 Z_1 + c_1 Z_2 && = d_1 \\
&a_2 Z_1 + b_2 Z_2 + c_2 Z_3 && = d_2 \\
&\qquad a_3 Z_2 + b_3 Z_3 + c_3 Z_4 && = d_3 \\
&\qquad\qquad \cdot && \cdot \\
&\qquad\qquad\quad \cdot && \cdot \\
&\qquad\qquad\qquad \cdot && \cdot \\
&\qquad\qquad\qquad a_m Z_{m-1} + b_m Z_m && = d_m
\end{aligned}
$$

where

$$Z_i = \theta_{j,\, i+1,\, n}$$

$$m = \text{KMAX} - 1$$

$$b_1 = \beta + \gamma + \rho\epsilon$$

$$c_1 = c_2 = \ldots = c_{m-1} = -\rho\epsilon$$

$$d_1 = \beta\theta_{j,\, 2,\, n-1} + \gamma\theta_{j-1,\, 2,\, n} - \rho(\Delta y_2)(-1)$$

$$a_2 = a_3 = \ldots = a_m = -\rho$$

$$b_2 = b_3 = \ldots = b_m = \beta + \gamma + \rho(1 + \epsilon)$$

$$d_i(i=2, 3, \ldots m) = \beta\theta_{j,\, i+1,\, n-1} + \gamma\theta_{j-1,\, i+1,\, n}$$

$\theta_{j,\, 1,\, n}$ is obtained by applying the boundary condition

$$\frac{1}{\Delta y_2}(\theta_{j,\, 2,\, n} - \theta_{j,\, 1,\, n}) = -1$$

where

$$\theta_{j,\, 1,\, n} = \theta_{j,\, 2,\, n} - \Delta y_2(-1) \tag{B3}$$

APPENDIX C

SOLUTION OF TRIDIAGONAL MATRIX EQUATION

The system of m tridiagonal equations (eq. (26)) can be solved by using Gauss' Elimination to yield the results

$$Z_m = d_m^*$$

$$Z_i = d_i^* - c_i^* Z_{i+1} \qquad i = m - 1,\ m - 2,\ \ldots 2, 1$$

where

$$c_1^* = \frac{c_1}{b_1} \qquad d_1^* = \frac{d_1}{b_1}$$

$$\left.\begin{aligned} c_i^* &= \frac{c_i}{b_i - a_i c_{i-1}^*} \\ d_i^* &= \frac{d_i - a_i d_{i-1}^*}{b_i - a_i c_{i-1}^*} \end{aligned}\right\} \quad i = 2, 3, \ldots m$$

and c_i^* and $d_i^*(i = 1, 2, \ldots m)$ are computed first and then used to find $Z_i(i = m,\ m - 1,\ \ldots 1)$.

If $(b_i - a_i c_{i-1}^*) = 0$, c_i^* and d_i^* are not computed but rather

$$d_{i+1}^* = \frac{d_i - a_i d_{i-1}^*}{c_i}$$

$$c_{i+1}^* = 0$$

$$Z_{i+1} = d_{i+1}^*$$

and

$$Z_i = \frac{d_{i+1} - b_{i+1} Z_{i+1} - c_{i+1} Z_{i+2}}{a_{i+1}}$$

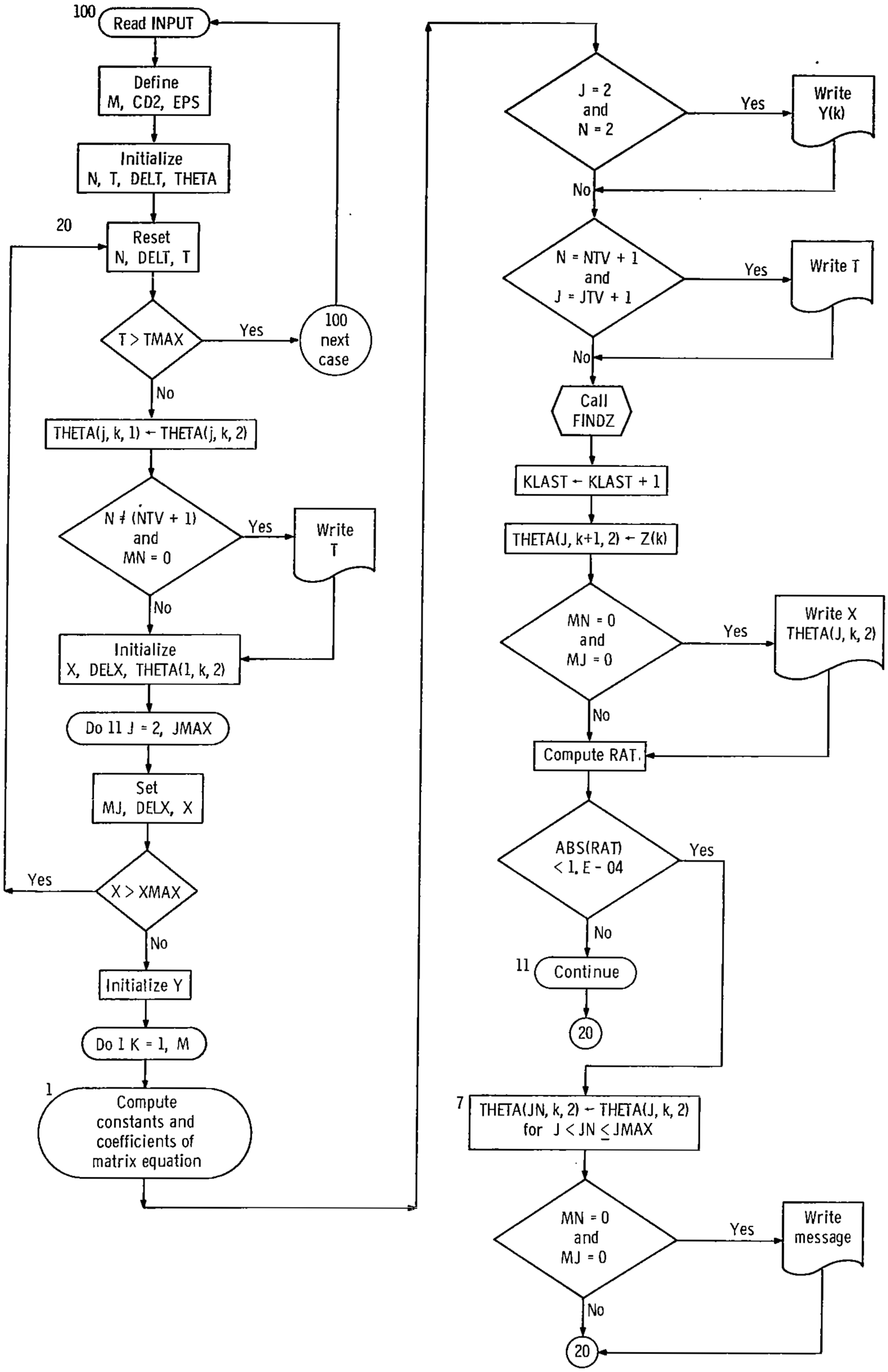


Figure 10. - Flow chart.

APPENDIX D

COMPUTER PROGRAM

The following sections give a detailed description of the computer program written to solve equation (24) numerically. The program consists of a main program and a subroutine FINDZ. Included herein are the general description of the program, a dictionary of the FORTRAN symbols used, a flow diagram (fig. 10), and the program listings. The input/output format is described, and the computer input/output of a sample case is also presented.

Program Description

Main program. - The input data are read, and the computation is begun by initializing x, y, t and inserting the boundary and initial conditions. For each subsequent row in x, the distance along x is incremented, the constants and coefficients of its matrix equation are calculated, and subroutine FINDZ is called.

Subroutine FINDZ. - The root of the matrix equation Z is found by the method described in appendix C. The column matrix Z is returned to the main program through blank COMMON.

Dictionary of FORTRAN Symbols

FORTRAN symbol	Engineering symbol	
X	$\bar{x} - \bar{x}_o$	distance along plate $-\bar{x}_o$
Y	y	array containing lattice dimensions in y-direction
T	t	time
XMAX		maximum x of interest
JMAX		number of lattice points in x-direction
KMAX		number of lattice points in y-direction
DELX	Δx_j	length of J^{th} increment in x
DELY	Δy_k	length of K^{th} increment in y
DELT	Δt_n	length of N^{th} increment in t

FORTRAN symbol	Engineering symbol	
DELXO		value of 1./PXINT times the first x-increment
DELYO		value of 1./PYINT times the first y-increment
DELTO		value of 1./PTINT times the first t-increment
THETA(J, K, 2)	$\theta_{j,k,n}$	evaluated at N^{th} interval in time
THETA(J, K, 1)	$\theta_{j,k,n-1}$	at the N^{th} interval in time
PARTYO		partial derivative of θ with respect to y at $y = 0$
RNU	ν	kinematic viscosity, input constant
ALPHA	α	thermal diffusivity, input constant
BETA	β	$1./\Delta t_n$
GAMMA	γ	$U_\infty \mathrm{erf}\, \eta/\Delta x_j$
SQ		$2\sqrt{\nu t}$
CD2		$(\alpha/\Delta y_2)$ times PARTYO
C2		variable equal to 0 or CD2, used in computing the D-array
UINF	U_∞	free-stream velocity, input constant
EPS	ϵ	1./PYINT
ERFARG	η	$y/2\sqrt{\nu t}$
FF		array containing values of $U_\infty \mathrm{erf}(\eta)$ at all y-intervals
M	m	KMAX - 1
A, B, C, D	a, b, c, d	arrays containing coefficients and constants of eq. (26)
CS(I)	c_i^*	defined in appendix C
DS(I)	d_i^*	defined in appendix C
DENOM		B(I) - (A(I) times CS(I - 1))
ITAG(I)		I^{th} element of array ITAG, an indicator of whether DENOM is zero; 0-no, and 1-yes
Z		array output from subroutine FINDZ, Z(K) = THETA(J, K+1, 2)

KLAST — on exit from FINDZ, KLAST is last nonzero element of Z; KLAST is immediately set equal to KLAST + 1 denoting the last nonzero element of THETA(J, K, 2) at given J

JTV, NTV — because of large volume of results, solution at every lattice point was not written out; solutions were written out only at every $(NTV)^{th}$ interval in time and $(JTV)^{th}$ interval in x

MN — result of MOD(N-1, NTV)

MJ — result of MOD(J-1, JTV)

RAT — result of 1.-THETA(J-1,KLAST,2)/THETA(J,KLAST,2); at small values of time, when x becomes greater than some specific X(J), dependent on value of T, THETA no longer changes significantly with x; when RAT is sufficiently small, THETA(JN, K, 2) is set equal to THETA(J, K, 2) for all values of K and for all $J < JN \leq JMAX$

INPUT — name list name containing values of ALPHA, RNU, UINF, PARTYO, XMAX, TMAX, JMAX, KMAX, DELXO, DELYO, DELTO, PXINT, PYINT, NTV, JTV

Blank COMMON contains M, A, B, C, D, Z, KLAST

The program listing is as follows:

```
C     ****IMPLICIT FINITE DIFFERENCE EQUATION ***********************
C  ***************IRREGULAR LATTICE SIZES  *************************************
C     RESULTS WRITTEN OUT ONLY AT EVERY NTV INTERVALS OF T AND EVERY JTV
C     INTERVALS OF X
      COMMON M,A,B,C,D,Z,KLAST
      DIMENSION A(100), B(100), C(100), D(100), Z(100), Y(100), THETA(12
     11, 100,2), FF(100)
      NAMELIST/INPUT/ ALPHA,RNU,UINF,PARTYO,XMAX,TMAX,JMAX,KMAX,DELXO,
     1DELYO,DELTO,PXINT,PYINT,PTINT,NTV,JTV
  100 READ(5,INPUT)
      M=KMAX - 1
      Y(1) = 0.
      CD2= -ALPHA * PARTYO/(DELYO*PYINT)
      EPS = 1./PYINT
      N= 1
      T = 0.
      DELT = DELTO
      DO 3 K=1,KMAX
      DO 3 J=1,JMAX
    3 THETA(J,K,2) = 0.
C     AT T=0., THETA=0. FOR ALL X AND Y
   20 N = N + 1
      DELT = DELT * PTINT
      T = T + DELT
      IF (T .GT. TMAX ) GO TO 100
```

```
      DO 10 J=1,JMAX
      DO 10 K=1,KMAX
   10 THETA(J,K,1) = THETA(J,K,2)
C     THETA(J,K,2) STANDS FOR THETA(J,K,N), THETA(J,K,1) FOR THETA(J,K,N-1)
      MN = MOD(N-1,NTV)
      IF ( N .NE. (NTV+1) .AND. MN .EQ. 0) WRITE(6,64) T
      BETA = 1./DELT
      SQ = 2.* SQRT(RNU*T)
      X = 0.
      DELX = DELXO
      DO 12 K=1,KMAX
   12 THETA(1,K,2) = 0.
C     AT X = 0., THETA=0.  FOR ALL Y AND T
      DO 11 J=2,JMAX
      MJ = MOD(J-1,JTV)
      DELX = PXINT * DELX
      X = X + DELX
      IF (X  .GT. XMAX) GO TO 20
C        X AND T ARE NOW FIXED, COMPUTE COEFFICIENTS AND CONSTANTS
C        OF FINITE-DIFFERENCE EQUATION AT EACH POINT OF Y
      DELY = DELYO
      DO 1 K=1,M
      IF (J .GT. 2) GO TO 2
      IF (N .GT. 2) GO TO 4
C     DELY, A, C, AND Y  ARE INDEPENDENT OF X AND T,  COMPUTED ONLY ONCE  IN
      THE PROGRAM, AT J=2, N=2
C     FF(K)  IS INDEPENDENT OF X, COMPUTED ONLY WHEN J=2 AT EVERY VALUE OF N
      DELY = PYINT * DELY
      ROE= ALPHA/DELY**2
      Y(K+1) = Y(K) +DELY
      A(K) = -ROE
      C(K) = -ROE*EPS
    4 ERFARG = Y(K+1)/SQ
      FF(K)= ERF(ERFARG)*UINF
    2 GAMMA = FF(K)/DELX
      B(K) = BETA + GAMMA - (A(K) +C(K) )
      D2 = 0.
      IF (K-1) 6,6,1
    6 D2 = CD2
      B(K) = BETA + GAMMA - C(K)
      A(K) = 0.
    1 D(K) = BETA * THETA(J,K+1,1) +GAMMA*THETA(J-1, K+1, 2) + D2
      C(M) = 0.
      IF (J .EQ. 2 .AND. N  .EQ. 2) WRITE(6,66)  (Y(K),K=1,KMAX)
      IF ( N .EQ. (NTV+1) .AND. J .EQ. (JTV+1) ) WRITE(6,63) T
      CALL FINDZ
C     KLAST IS THE  LAST NONZERO ELEMENT OF THE SOLUTION FROM FINDZ
      KLAST = KLAST+1
      DO 5 K=1,M
    5 THETA(J,K+1,2) = Z(K)
      THETA(J,1,2) = THETA(J,2,2) - (PYINT*DELYO) * PARTYO
      IF(MN .EQ. 0 .AND. MJ .EQ. 0)WRITE(6,65)X,(THETA(J,K,2),K=1,KLAST)
   13 RAT = 1. - THETA(  J-1, KLAST,2)/ THETA(  J, KLAST, 2)
      IF (ABS(RAT) .LT. 1.E-04) GO TO 7
C     IF THETA NO LONGER CHANGE WITH X, JUMP OUT OF DO 11 LOOP
   11 CONTINUE
      GO TO 20
    7 JP = J + 1
      DO 8 JN = JP,JMAX
      DO 8 K=1,KMAX
    8 THETA(JN, K, 2) = THETA(J  ,K,2)
      IF ( MN .EQ. 0 ) WRITE(6,67) X
      GO TO 20
```

Program Listing (C)

```
 63   FORMAT(1HL,55X, 9HAT TIME =, F8.5, 2X, 4HSEC.)
 64   FORMAT(1H1,55X, 9HAT TIME =, F8.5, 2X, 4HSEC.)
 65   FORMAT( / ,30X, 6HAT X =, E13.5, 3H CM, 2X,      59HTHETA AT VARIO
     1US VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED),/ (10(1PE13.4)))
 66   FORMAT(1H1, 45X, 41HVALUES OF Y AT WHICH THETA ARE CALCULATED,/
     1(10(1PE13.4)))
 67   FORMAT(1HL, 30X, 48HTHETA NO LONGER CHANGES WITH X AT X GREATER TH
     1AN,E13.5, 3H CM)
      END
```

```
$IBFTC TRID
      SUBROUTINE FINDZ
C     SOLUTION OF TRIDIAGONAL MATRIX EQUATION BY GAUSS' ELIMINATION
C     REFERENCE -- FORSYTHE AND WASOW P.104
      DIMENSION A(100),B(100),C(100),D(100),Z(100),CS(100),DS(100),
     1ITAG(100)
      COMMON M,A,B,C,D,Z,KLAST
      ITAG(1) = 0
      CS(1) = C(1)/B(1)
      DS(1) = D(1) /B(1)
      DO 4 I=2,M
      ITAG(I) =0
      IF (ITAG(I-1) .EQ. 1) GO TO 4
      DENOM = B(I) - A(I)*CS(I-1)
      IF (ABS(DENOM) - 1.E-37) 1,1,3
 1    ITAG(I) = 1
C     DENOM(I) = 0.  NO CS(I) AND NO DS(I) COMPUTED
      DS(I+1) = (D(I)-A(I)*DS(I-1))/C(I)
      CS(I+1) = 0.
      GO TO 4
 3    DS(I) = (D(I) - A(I)*DS(I-1) )/DENOM
      CS(I) = C(I)/DENOM
 4    CONTINUE
      DO 5 I=1,M
      K=M-I+1
      IF (ABS(DS(K)) .GT. 1.E-37) GO TO 6
 5    Z(K) = 0.
 6    Z(K) = DS(K)
      KLAST = K
C     KLAST IS THE LAST NON-ZERO Z ELEMENT
      KM1 = K-1
      DO 9 J=1,KM1
      I=K-J
      IF (ITAG(I)) 7,7,8
 7    Z(I) = DS(I) - CS(I)*Z(I+1)
      GO TO 9
 8    Z(I) = (D(I+1) -B(I+1)*Z(I+1) -C(I+1)*Z(I+2) )/A(I+1)
 9    CONTINUE
      RETURN
      END
```

Description of Input and Output

Input. - Input data are read into the program in namelist form. A namelist statement in the main program specifies a list of variables under the namelist name INPUT. Data cards must be written in the form appropriate for namelist input for the computer used. For the present case, the computer used is the IBM 7090/7040 direct-couple system, and the input card format is specified in reference 14. The first card of each case must have a $ sign in the second column followed immediately by the namelist name. Each input variable is given in the following form: variable name = number, separated by commas. A $ sign must follow the last data item indicating the end of a namelist data group. In the program, the system of units used for input/output is not restricted. In the input data cards for the following sample case, all the values of the dimensioned physical variables are given in cgs-centigrade units: ALPHA and RNU in square centimeters per second, UINF in centimeters per second, XMAX, DELXO, and DELYO in centimeters, and TMAX and DELTO in seconds. All other input variables are dimensionless quantities. The following listing shows an input data card of a sample case:

```
$INPUT ALPHA=1.6722E-03, RNU=3.1586E-03, UINF=3.048, PARTYO=-1.0,
   XMAX=5.0, TMAX=1.E-04, JMAX=121, KMAX=100, DELXO=5.5E-09, DELYO=4.5E-06,
   DELTO=1.E-05, PXINT=1.2, PYINT=1.1, PTINT=1.1, NTV=3, JTV=12$
```

Output. - The lattice positions in the y-direction are written out at the beginning of the computation. Values of THETA at these y-positions will be written out at increasing values of x at every NTVth value of t.

In some instances, one may wish to see the results in terms of $T - T_\infty$ instead of θ defined presently as $(T - T_\infty)/(Q_W/Ak)$. Rather than multiplying the final result at each point by the constant (Q_W/Ak), it is more expedient to redefine θ as $T - T_\infty$. The differential equation will remain the same. The only alteration required in the program is the input $\partial\theta/\partial y$ at $y = 0$(PARTYO), which, with the new definition of θ, has to be multiplied by (Q_W/Ak). The following listing shows the computer output of a sample case:

VALUES OF Y AT WHICH THETA ARE CALCULATED

0.	4.9500E-06	1.0395E-05	1.6384E-05	2.2973E-05	3.0220E-05	3.8192E-05	4.6961E-05	5.6608E-05	6.7218E-05
7.8890E-05	9.1729E-05	1.0585E-04	1.2139E-04	1.3848E-04	1.5727E-04	1.7795E-04	2.0070E-04	2.2572E-04	2.5324E-04
2.8351E-04	3.1681E-04	3.5344E-04	3.9374E-04	4.3806E-04	4.8682E-04	5.4045E-04	5.9944E-04	6.6434E-04	7.3572E-04
8.1425E-04	9.0062E-04	9.9563E-04	1.1001E-03	1.2151E-03	1.3416E-03	1.4807E-03	1.6337E-03	1.8020E-03	1.9872E-03
2.1908E-03	2.4149E-03	2.6613E-03	2.9324E-03	3.2306E-03	3.5586E-03	3.9194E-03	4.3163E-03	4.7529E-03	5.2331E-03
5.7613E-03	6.3424E-03	6.9816E-03	7.6847E-03	8.4582E-03	9.3089E-03	1.0245E-02	1.1274E-02	1.2407E-02	1.3652E-02
1.5022E-02	1.6530E-02	1.8187E-02	2.0011E-02	2.2017E-02	2.4224E-02	2.6651E-02	2.9321E-02	3.2258E-02	3.5489E-02
3.9043E-02	4.2952E-02	4.7252E-02	5.1983E-02	5.7186E-02	6.2909E-02	6.9205E-02	7.6131E-02	8.3749E-02	9.2128E-02
1.0135E-01	1.1149E-01	1.2264E-01	1.3491E-01	1.4840E-01	1.6325E-01	1.7958E-01	1.9754E-01	2.1730E-01	2.3904E-01
2.6294E-01	2.8924E-01	3.1817E-01	3.5000E-01	3.8500E-01	4.2351E-01	4.6586E-01	5.1245E-01	5.6370E-01	6.2008E-01

AT TIME = 0.00004 SEC.

AT X = 0.26123E-06 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

6.7824E-05	6.2874E-05	5.7472E-05	5.1633E-05	4.5398E-05	3.8852E-05	3.2134E-05	2.5446E-05	1.9061E-05	1.3298E-05
8.4738E-06	4.8164E-06	2.3762E-06	9.8811E-07	3.3636E-07	9.1274E-08	1.9312E-08	3.1335E-09	3.8546E-10	3.5701E-11
2.4816E-12	1.2941E-13	5.0716E-15	1.4985E-16	3.3533E-18	5.7155E-20	7.4677E-22	7.5341E-24	5.9156E-26	3.6453E-28
1.7783E-30	6.9267E-33	2.1715E-35							

AT X = 0.25904E-05 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

1.4218E-04	1.3723E-04	1.3181E-04	1.2588E-04	1.1943E-04	1.1242E-04	1.0484E-04	9.6685E-05	8.7985E-05	7.8785E-05
6.9175E-05	5.9292E-05	4.9336E-05	3.9569E-05	3.0316E-05	2.1940E-05	1.4789E-05	9.1294E-06	5.0592E-06	2.4611E-06
1.0263E-06	3.5837E-07	1.0256E-07	2.3633E-08	4.3260E-09	6.2363E-10	7.0489E-11	6.2425E-12	4.3412E-13	2.3816E-14
1.0369E-15	3.6083E-17	1.0108E-18	2.2953E-20	4.2498E-22	6.4466E-24	8.0379E-26	8.2548E-28	6.9909E-30	4.8851E-32
2.8168E-34	1.3216E-36								

AT X = 0.23357E-04 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

2.4571E-04	2.4076E-04	2.3536E-04	2.2950E-04	2.2313E-04	2.1623E-04	2.0876E-04	2.0070E-04	1.9201E-04	1.8269E-04
1.7271E-04	1.6209E-04	1.5082E-04	1.3895E-04	1.2653E-04	1.1364E-04	1.0041E-04	8.6996E-05	7.3624E-05	6.0558E-05
4.8118E-05	3.6655E-05	2.6524E-05	1.8031E-05	1.1367E-05	6.5478E-06	3.3926E-06	1.5555E-06	6.2141E-07	2.1335E-07
6.2254E-08	1.5313E-08	3.1585E-09	5.4454E-10	7.8344E-11	9.3967E-12	9.3881E-13	7.8051E-14	5.3939E-15	3.0949E-16
1.4728E-17	5.8084E-19	1.8969E-20	5.1271E-22	1.1464E-23	2.1200E-25	3.2412E-27	4.0961E-29	4.2784E-31	3.6929E-33
2.6321E-35	1.5045E-37								

AT X = 0.20852E-03 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

2.7582E-04	2.7087E-04	2.6550E-04	2.5968E-04	2.5338E-04	2.4658E-04	2.3926E-04	2.3138E-04	2.2295E-04	2.1394E-04
2.0436E-04	1.9420E-04	1.8349E-04	1.7225E-04	1.6054E-04	1.4840E-04	1.3594E-04	1.2326E-04	1.1047E-04	9.7738E-05
8.5220E-05	7.3097E-05	6.1552E-05	5.0764E-05	4.0897E-05	3.2089E-05	2.4437E-05	1.7992E-05	1.2748E-05	8.6459E-06
5.5791E-06	3.4010E-06	1.9429E-06	1.0309E-06	5.0303E-07	2.2341E-07	8.9338E-08	3.1816E-08	9.9835E-09	2.7319E-09
6.4575E-10	1.3072E-10	2.2492E-11	3.2678E-12	3.9873E-13	4.0673E-14	3.4558E-15	2.4384E-16	1.4255E-17	6.8925E-19
2.7525E-20	9.0691E-22	2.4636E-23	5.5150E-25	1.0170E-26	1.5446E-28	1.9319E-30	1.9899E-32	1.6875E-34	1.1031E-36

AT X = 0.18594E-02 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

2.7593E-04	2.7098E-04	2.6560E-04	2.5978E-04	2.5349E-04	2.4669E-04	2.3936E-04	2.3149E-04	2.2306E-04	2.1406E-04
2.0447E-04	1.9432E-04	1.8361E-04	1.7238E-04	1.6067E-04	1.4854E-04	1.3609E-04	1.2342E-04	1.1065E-04	9.7928E-05
8.5429E-05	7.3327E-05	6.1806E-05	5.1044E-05	4.1206E-05	3.2426E-05	2.4800E-05	1.8374E-05	1.3140E-05	9.0336E-06
5.9450E-06	3.7275E-06	2.2153E-06	1.2412E-06	6.5169E-07	3.1867E-07	1.4417E-07	5.9921E-08	2.2717E-08	7.7978E-09
2.4052E-09	6.6158E-10	1.6107E-10	3.4458E-11	6.4318E-12	1.0405E-12	1.4498E-13	1.7300E-14	1.7587E-15	1.5161E-16
1.1037E-17	6.7603E-19	3.4732E-20	1.4926E-21	5.3526E-23	1.5986E-24	3.9693E-26	8.1816E-28	1.3982E-29	1.9792E-31
2.3183E-33	2.2451E-35	1.7677E-37							

THETA NO LONGER CHANGES WITH X AT X GREATER THAN 0.55523E-02 CM

AT TIME = 0.00008 SEC.

AT X = 0.26123E-06 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

7.8279E-05	7.3329E-05	6.7915E-05	6.2035E-05	5.5708E-05	4.8984E-05	4.1954E-05	3.4767E-05	2.7633E-05	2.0829E-05
1.4677E-05	9.4959E-06	5.5188E-06	2.8087E-06	1.2174E-06	4.3672E-07	1.2623E-07	2.8721E-08	5.0486E-09	6.7627E-10
6.8393E-11	5.1929E-12	2.9521E-13	1.2558E-14	4.0019E-16	9.5745E-18	1.7254E-19	2.3514E-21	2.4349E-23	1.9262E-25
1.1712E-27	5.5096E-30	2.0201E-32	5.8153E-35	1.3040E-37					

AT X = 0.25904E-05 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

1.6655E-04	1.6160E-04	1.5617E-04	1.5022E-04	1.4371E-04	1.3660E-04	1.2888E-04	1.2051E-04	1.1151E-04	1.0187E-04
9.1658E-05	8.0952E-05	6.9895E-05	5.8692E-05	4.7617E-05	3.7018E-05	2.7289E-05	1.8829E-05	1.1968E-05	6.8760E-06
3.4935E-06	1.5326E-06	5.6652E-07	1.7235E-07	4.2267E-08	8.2162E-09	1.2503E-09	1.4772E-10	1.3493E-11	9.5165E-13
5.1908E-14	2.1976E-15	7.2591E-17	1.8830E-18	3.8642E-20	6.3218E-22	8.3095E-24	8.8390E-26	7.6573E-28	5.4302E-30
3.1638E-32	1.5176E-34	5.9151E-37							

AT X = 0.23357E-04 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

3.2403E-04	3.1908E-04	3.1365E-04	3.0771E-04	3.0121E-04	2.9409E-04	2.8632E-04	2.7783E-04	2.6859E-04	2.5854E-04
2.4763E-04	2.3581E-04	2.2307E-04	2.0937E-04	1.9472E-04	1.7915E-04	1.6271E-04	1.4554E-04	1.2780E-04	1.0975E-04
9.1740E-05	7.4202E-05	5.7648E-05	4.2632E-05	2.9682E-05	1.9201E-05	1.1366E-05	6.0504E-06	2.8438E-06	1.1583E-06
4.0175E-07	1.1686E-07	2.8171E-08	5.5818E-09	9.0484E-10	1.1982E-10	1.2971E-11	1.1504E-12	8.3818E-14	5.0299E-15
2.4909E-16	1.0192E-17	3.4471E-19	9.6388E-21	2.2280E-22	4.2566E-24	6.7207E-26	8.7685E-28	9.4528E-30	8.4198E-32
6.1962E-34	3.7186E-36								

AT X = 0.20852E-03 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

4.2695E-04	4.2200E-04	4.1660E-04	4.1071E-04	4.0430E-04	3.9732E-04	3.8973E-04	3.8150E-04	3.7259E-04	3.6294E-04
3.5253E-04	3.4132E-04	3.2927E-04	3.1637E-04	3.0259E-04	2.8794E-04	2.7241E-04	2.5604E-04	2.3889E-04	2.2102E-04
2.0254E-04	1.8359E-04	1.6435E-04	1.4504E-04	1.2590E-04	1.0722E-04	8.9316E-05	7.2502E-05	5.7100E-05	4.3399E-05
3.1632E-05	2.1942E-05	1.4360E-05	8.7770E-06	4.9556E-06	2.5545E-06	1.1879E-06	4.9262E-07	1.8016E-07	5.7522E-08
1.5885E-08	3.7624E-09	7.5852E-10	1.2930E-10	1.8529E-11	2.2214E-12	2.2191E-13	1.8413E-14	1.2656E-15	7.1923E-17
3.3740E-18	1.3051E-19	4.1588E-21	1.0911E-22	2.3560E-24	4.1857E-26	6.1180E-28	7.3567E-30	7.2777E-32	5.9233E-34
3.9390E-36									

AT X = 0.18594E-02 CM THETA AT VARIOUS VERTICAL POINTS (NONUNIFORMLY DISTRIBUTED)

4.2860E-04	4.2365E-04	4.1825E-04	4.1236E-04	4.0595E-04	3.9897E-04	3.9140E-04	3.8318E-04	3.7427E-04	3.6464E-04
3.5426E-04	3.4307E-04	3.3106E-04	3.1820E-04	3.0447E-04	2.8988E-04	2.7443E-04	2.5816E-04	2.4112E-04	2.2339E-04
2.0507E-04	1.8631E-04	1.6729E-04	1.4822E-04	1.2935E-04	1.1095E-04	9.3331E-05	7.6785E-05	6.1600E-05	4.8025E-05
3.6249E-05	2.6376E-05	1.8415E-05	1.2272E-05	7.7622E-06	4.6310E-06	2.5887E-06	1.3461E-06	6.4627E-07	2.8420E-07
1.1354E-07	4.0864E-08	1.3139E-08	3.7426E-09	9.3682E-10	2.0446E-10	3.8620E-11	6.2704E-12	8.6961E-13	1.0243E-13
1.0196E-14	8.5373E-16	5.9904E-17	3.5103E-18	1.7129E-19	6.9432E-21	2.3331E-22	6.4876E-24	1.4908E-25	2.8279E-27
4.4239E-29	5.7032E-31	6.0559E-33	5.2891E-35	3.6797E-37					

THETA NO LONGER CHANGES WITH X AT X GREATER THAN 0.66628E-02 CM

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